Mini Homework 2
Math 117 - Summer 2022

1) (2 points) Let S be a subset of a vector space V. Show that $span(S)$ is the smallest subspace containing all the vectors in S.

Solution:

2) (1 point) Let V be a vector space over some field $\mathbb{F}$ and let $L = \{v_1, v_2, \ldots, v_k\} \subset V$ be a finite linearly independent ordered set. Suppose $w \in span(L)$. Then w is expressed **uniquely** as a linear combination of elements in L.

Solution:

3) Let V, W be finite dimensional vector spaces and let $\mathcal{B} = \{v_1, \ldots, v_n\}$ be a basis for V. Let $T : V \to W$ be a linear map.

(a) (2 points) Show that T is surjective $\iff$ $T(\mathcal{B})$ spans W

(b) (1 point) Conclude that T is an isomorphism $\iff$ $T(\mathcal{B})$ is a basis for W

Solution:

4) Consider the following three vector spaces with corresponding basis:

$$
\begin{aligned}
V_1 &= \mathbb{R}^3 & \mathcal{B}_1 &= \{e_1, e_2, e_3\} \\
V_2 &= \mathbb{R}[t]_{\le 2} & \mathcal{B}_2 &= \{1, t, t^2\} \\
V_3 &= M_{2\times 2}(\mathbb{R}) & \mathcal{B}_3 &= \{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\}
\end{aligned}
$$

Now consider the following two linear transformations:

$$T\left(\begin{pmatrix} a \\ b \\ c \end{pmatrix}\right) = a + b - ct + at^2$$

$$S(a_0 + a_1 t + a_2 t^2) = \begin{pmatrix} a_0 - a_2 & a_1 \\ a_0 + a_1 & a_2 \end{pmatrix}$$

(a) (1 point) Compute $[T]_{\mathcal{B}_1}^{\mathcal{B}_2}$

(b) (1 point) Compute $[S]_{\mathcal{B}_2}^{\mathcal{B}_3}$

(c) (2 points) Compute $[S \circ T]_{\mathcal{B}_1}^{\mathcal{B}_3}$ and verify that $[S \circ T]_{\mathcal{B}_1}^{\mathcal{B}_3} = [S]_{\mathcal{B}_2}^{\mathcal{B}_3}[T]_{\mathcal{B}_1}^{\mathcal{B}_2}$